Surds equations

(1) **(a)** Squaring creates roots.

(b) Beware of hidden givens.

(c) Check your roots.

Example 1

Solve : $\sqrt{3x^2 - 4x + 5} = 2x - 3$ for real roots.

Solution Squaring the given equation, we have:

$$\begin{cases} 3x^2 - 4x + 5 = (2x-3)^2 & (1) \\ 3x^2 - 4x + 5 \geq 0 & (2) \\ 2x - 3 \geq 0 & (3) \end{cases}$$

Note that (2) and (3) are hidden givens .

From (1), $x^2 - 8x + 4 = 0 \quad \Rightarrow \quad x = 4 \pm \sqrt{3}$

$x = 4 - \sqrt{3}$ does not satisfy (3) since $2x - 3 = 2(4 - \sqrt{3}) - 3 = 5 - 4\sqrt{3} < 0$

$\therefore$ $x = 4 - \sqrt{3}$ is rejected.

$x = 4 + \sqrt{3}$ satisfy both (2) and (3) .

(Ans) $x = 4 + \sqrt{3}$ is the only root.

Example 2

Solve : $(x^2 + x)^2 + \sqrt{x^2 - 1} = 0$ for real roots.

Solution From the given equation :

$$\begin{cases} x^2 + x = 0 & (1) \\ x^2 - 1 = 0 & (2) \end{cases}$$

Solving, the only root is $x = -1$.

(2) In some surds equations, **transpose term** before squaring may be better.

Example 3

Solve : $\sqrt{2x+1} + \sqrt{2x-1} = 2$ for real roots.

Solution Transpose term in the given equation : $\sqrt{2x+1} = 2 - \sqrt{2x-1}$

Squaring, $2x + 1 = 4 - 4\sqrt{2x-1} + 2x - 1$

$4\sqrt{2x-1} = 2$

$2\sqrt{2x-1} = 1$

Squaring again, $4(2x - 1) = 1$

$\therefore$ $x = \dfrac{5}{8}$, which is a good root on checking .

Exercise Solve $\sqrt{4x+3} - \sqrt{4x-1} = 2$ for real roots. **Ans :** $\dfrac{1}{4}$.

Example 4

Solve : $\sqrt{5x+1}-\sqrt{x}=2$ for real roots.

Solution Transpose term in the given equation : $\sqrt{5x+1}=2+\sqrt{x}$.

Squaring, $5x+1=x+4\sqrt{x}+4$

$4\sqrt{x}=4x-3$

Squaring again, $16x = 16x^2 - 24x + 9$

$16x^2 - 40x + 9 = 0$

$(4x - 9)(4x - 1) = 0$

$\therefore \quad x=\frac{9}{4}$ or $\frac{1}{4}$

On checking, $x=\frac{1}{4}$ is a redundant root and should be rejected. $\therefore \quad x=\frac{9}{4}$

Exercise Solve $\sqrt{x+1}-\sqrt{1-2x}=1$ for real roots. **Ans :** $\frac{2\sqrt{7}-1}{9}$

(3) Conjugate method

Example 4

Solve : $\sqrt{3x+4}+\sqrt{3x-6}=10$ for real roots.

Solution $\sqrt{3x+4}+\sqrt{3x-6}=10$ …. (1)

Let $\sqrt{3x+4}-\sqrt{3x-6}=y$ …. (2)

(1)×(2), $(3x+4)-(3x-6)=10y$

$\therefore \quad y = 1$.

From (2), $\sqrt{3x+4}-\sqrt{3x-6}=1$ …. (3)

(1) + (3), $2\sqrt{3x+4}=11$

Squaring, $4(3x + 4) = 121$

$\therefore \quad x=\frac{35}{4}$. (on checking, this is a good root)

Example 5

Solve : $\sqrt{x^2+3x+7}+\sqrt{x^2+3x-9}=2$ for real roots.

Solution $\sqrt{x^2+3x+7}+\sqrt{x^2+3x-9}=2$ …. (1)

Let $\sqrt{x^2+3x+7}-\sqrt{x^2+3x-9}=y$ …. (2)

(1)×(2), $(x^2+3x+7)-(x^2+3x-9)=2y$

$\therefore \quad y = 8$.

From (2), $\sqrt{x^2+3x+7}-\sqrt{x^2+3x-9}=8$ …. (3)

[(1) + (3)]/2, $\sqrt{x^2+3x+7}=5$ …. (4)

Squaring and simplify, $x^2 + 3x - 18 = 0$

$$(x-3)(x+6)=0$$

$$\therefore \quad x = 3\,,-6$$

Since both sides of (4) are positive, we do not have any redundant root on squaring.

Exercise Solve $\sqrt{x^2-4x+34}+\sqrt{x^2-4x-11}=9$ for real roots. **Ans :** $3,-\frac{5}{3}$

(4) Change of variables

Example 6

Solve : $x^2-2x=\sqrt{2x^2-4x+3}$ for real roots.

Solution $x^2-2x=\sqrt{2x^2-4x+3}$ .... (1)

Put $y=x^2-2x$ .... (2)

then $2x^2-4x+3 = 2y+3$

The given equation then becomes $y=\sqrt{2y+3}$ .... (3)

Squaring , $y^2-2y-3=0$, $(y-3)(y+1)=0$

Since from (3), $y \ge 0$, $y+1 \ne 0$, $\therefore$ $y = 3$ .... (4)

(4)↓(2), $x^2-2x-3=0$, $(x-3)(x+1)=0$

$\therefore$ $x = 3, -1$.

Example 7

Solve : $\sqrt{x+2}-\sqrt{x-k+2}=1$, where k is a constant .

Solution Let $\begin{cases} u=\sqrt{x+2}\ge 0 \\ v=\sqrt{x-k+2}\ge 0 \end{cases}$

Then the given equation becomes : $\begin{cases} u-v=1 \\ u^2-v^2=k \end{cases}$

$$\Leftrightarrow \begin{cases} u-v=1 \\ (u+v)(u-v)=k \end{cases} \Leftrightarrow \begin{cases} u-v=1 \\ u+v=k \end{cases} \Leftrightarrow \begin{cases} u=\frac{1}{2}(k+1) \\ v=\frac{1}{2}(k-1) \end{cases}$$

Since $\sqrt{x+1}=\frac{1}{2}(k+1)\ge 0,\quad \sqrt{x-k+1}=\frac{1}{2}(k-1)\ge 0$

therefore $\begin{cases} x=\frac{1}{4}(k+1)^2-2 & ,\text{when } k\ge 1 \\ \text{no solution} & ,\text{when } k<1 \end{cases}$

Exercise Solve $\sqrt{x+1}-\sqrt{\frac{x-1}{x}}=1$ **Ans:** $x=\frac{1+\sqrt{5}}{2}$